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PERSONAL INCOME TAXATION AND THE PRINCIPLE OF EQUAL SACRIFICE REVISITED*

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This paper shows that, in the domain of piecewise linear statutory income tax functions, the principle of equal sacrifice implies tax progressivity. The progressivity implication of the doctrine is, in fact, stronger: the equal sacrifice principle, in essence, characterizes marginal rate progressivity, a result which is in sharp contrast with the standard literature on public finance. We also apply our findings to the personal statutory income taxation practices of the OECD countries and observe that the United States and Turkey were the only ones violating the principle of equal sacrifice in the time period 1988 to 1991.

The obviously equitable principle... is that equal sacrifices should be imposed on all.
The Principles of Political Economy, Henry Sidgwick.

1. INTRODUCTION

At the center of almost any debate on income taxation is the principle of tax progressivity. In their widely cited work, Blum and Kalven (1953, p. v) argue that "every controversy about changes in income tax rates is to some extent a controversy over the principle of progression itself." The principle envisages that the amount of income tax paid as a proportion of income should rise with income. The traditional rationale for it states that the rich should pay at a higher rate because the induced loss falls on them relatively more lightly.[2] Intuitive as it may be, it seems safe to say that this rationale is too informal to account for the fact that the income taxes in practice are almost exclusively progressive.[3,4]

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[2] See Lambert (1993) and Young (1994) for extensive examinations of this principle and for detailed accounts of the theory of progressive taxation.
[3] For instance, OECD (1981, 1986) report that all the personal income tax schedules of OECD countries are progressive. The difference of the taxation schemes appear, among other things, on the degree of tax progressivity.
[4] One should, however, note that a formal justification of the progressivity principle with respect to *inequality reduction* is given by the valuable contributions of Jakobsson (1976), Fellman (1976) and Kakwani (1977): An (incentive preserving) tax function reduces the relative inequality (in the sense of shifting the Lorenz curve of the pre-tax distribution upwards) for any given pre-tax income distribution if, and only if, it is progressive. Therefore, as also argued by Blum and Kalven (1953), progressive taxation can be seen as an insurance that the public demands to reduce inequality.

In accordance with the doctrine of ability to pay, a traditional approach to personal income taxation has attempted to justify the progressivity principle on the basis of the theories of equal sacrifice.[5] These hedonistic approaches to income taxation were not formalized until very recently, and a characterization of progressive taxation with respect to the time-honored principle of equal sacrifice has never been demonstrated. Moreover, since, to apply the principle, one needs the precise form of the utility function of each individual relative to which their sacrifices shall be equated and since such a practice assumes interpersonal comparability of individual preferences, these traditional approaches were rather forgotten with the emergence of the new welfare economics.

Starting from the late 1980s, however, we witness a noticeable revival of interest in equal sacrifice theories.[6] In particular, Young (1988) and Ok (1995) show that a number of compelling *ordinal* taxation properties imply the existence of a social norm (or, a utility function for the representative agent of the society) relative to which each agent sacrifices equally. Since 'sacrifice' is measured in terms of a single utility function (acting as a social norm) in these studies, the problem of interpersonal comparisons of preferences is avoided in a trivial way. Moreover, the cardinality of this utility function is not assumed at the outset, but instead derived as a consequence of some elementary taxation principles. Therefore, we can conclude that such a development of the theory of equal sacrifice is free of the standard criticisms. The real question concerns the usefulness of such an approach and asks if it is possible to relate the principle of equal sacrifice to the principle of progressivity.

This question was, of course, addressed in the earlier literature, and the answer is now widely recognized as a negative one. The argument is that "there is no ready basis on which to conclude whether equal absolute sacrifice calls for progression, not to speak of the proper degree of progression."[7] The common contention is that equal (absolute) sacrifice with respect to a given utility function implies progressive, proportional, or regressive taxation provided that the absolute value of the Arrow–Pratt coefficient of relative risk aversion is greater than, equal to, or less than unity. This conclusion, attributed to Samuelson (1947), is perhaps the most important reason why the ability-to-pay doctrine is widely held to be *inconclusive* with regard to the problem of tax design. Indeed, Samuelson (1947) renders the principle of equal sacrifice as 'conservative' on the basis of this observation and states that "following [equal sacrifice doctrine] we can only be sure that taxes should increase with income, but not necessarily in proportion with income."[8]

[5] See, for instance, Mill (1848), Sidgwick (1883), Cohen Stuart (1889) and Edgeworth (1897). See also Musgrave and Peacock (1958) and Musgrave (1985) for broader treatments in a historical context. Although critically scrutinized by well-known economists, as Pechmann (1990, p. 6) puts it, "the ability to pay idea has been a powerful source in history and has doubtedly contributed to the widespread acceptance of progressive taxation."

[6] See, for instance, Richter (1983), Buchholz et al. (1988), Young (1987, 1988, 1990), Yaari (1988), Berliant and Gouveia (1993) and Ok (1995).

[7] Musgrave and Musgrave (1980, p. 251).

[8] Samuelson (1947), pp. 226–227.

To prove his claim, Samuelson (1947, p. 227) starts with the fact that an equal (absolute) sacrifice tax function $t(\cdot)$ satisfies

$$(1) \qquad u(x) - u(x - t(x)) = \text{constant for all } x > 0$$

for some differentiable, strictly increasing and concave $u(\cdot)$, and upon differentiating and rearranging, concludes that $t(\cdot)$ is *not* progressive if, and only if,

$$(2) \qquad \frac{u'(x)x}{u'(x - t(x))(x - t(x))} > 1 \text{ for some } x > 0.$$

This demonstration is commonly taken to imply that a tax function $t(\cdot)$ satisfying (1) need not be progressive. But does this conclusion really follow from the above analysis? The answer would be no unless we can be sure that, given a nonprogressive $t(\cdot)$, both (1) and (2) can be simultaneously satisfied for at least one differentiable, strictly increasing and concave utility function. Without verifying the existence of such a utility function, we cannot be sure that Samuelson's proposition is valid. Indeed, if, given $t(\cdot)$, (1) cannot hold for any given utility function satisfying the stated properties, then $t(\cdot)$ *cannot* be an equal sacrifice tax to begin with. Consequently, whether a regressive tax function can be an equal sacrifice tax function or not is still an open question.

To give a precise answer to this question, one must of course specify the admissible classes of tax and utility functions. In this paper, we depart from the earlier approaches to tax equity assessment only in that we work with the class of piecewise linear tax functions as opposed to the class of all continuous tax functions. It must, however, be clear that this is not an unacceptable restriction in view of the fact that almost all countries in the world use (statutory) tax schedules specified only in terms of the tax brackets and the tax rates.[9] Assuming piecewise linear tax functions, our main results can be summarized as follows:

(i) An equal sacrifice tax function *cannot* be regressive.

(ii) Except for some pathological cases, an equal sacrifice tax function must be not only progressive, but convex (marginal rate progressive).[10]

(iii) Any marginal-rate progressive tax is an equal sacrifice tax (with respect to some strictly increasing and concave utility function).

Given these results, we argue that one has to reevaluate the widely held belief that the principle of equal sacrifice is inconclusive with regard to progressivity considerations, and further conclude that, for all practical purposes, the equal sacrifice doctrine does imply progressivity after all, a position that Mill (1848) so forcefully defended.

Another implication of our results concerns evaluating the actual income tax schedules. If any given progressive tax function were an equal sacrifice tax with

[9] One exception to this statement is supplied by Germany, which used a formula-based tax structure instead. (See OECD 1981.)

[10] The pathological cases constitute a negligible set in a measure theoretic sense. Maybe more important than this formality, we should note that none of the tax structures of the OECD countries turn out to be 'pathological.'

respect to an admissible utility function, then we could not learn much from a given progressive tax with regard to its sacrifice equitability. For, in that case, all that our theory could conclude would be that there exists some well-behaved utility function relative to which sacrifices are equated, but there is, of course, no way of checking whether this hypothetical social norm reflects the true preferences of the constituents of the society. But, it is possible to learn more from the theory (especially from result ii noted above). Indeed, we now know that a progressive income tax function does not inflict equal sacrifice upon all for any utility function (or better, for any representative agent) unless it is convex, that is, marginal-rate progressive. Consequently, a nonconvex income tax function has to be rejected by the ability to pay rule, for such a function cannot be an equal sacrifice tax no matter what the true preferences of the individuals are.[11] Therefore, we learn that a nonconvex income tax function must be rendered *inequitable* with respect to the doctrine of ability to pay. It is interesting to observe that only the OECD countries that applied nonconvex progressive income tax schedules in the period of 1988–1991 are the United States and Turkey, and therefore, among all the OECD members, these two countries are the only ones which violated the principal of equal sacrifice in income taxation with certainty during this period.

The structure of the paper is as follows. Section 2 introduces the basic definitions and formally states the problem at hand. In Section 3, we show that any marginally progressive income tax function can, in fact, be regarded as an equal sacrifice tax. This section also determines what class of tax functions can be viewed as supported by the ability to pay doctrine. Moreover, in this section, we show that the converse of the previous statement is essentially true, and compare our conclusions to a fundamental characterization result established in the theory of inequality reducing redistribution. Section 4 attempts to evaluate the U.S. federal income taxation practice from an equal sacrifice perspective, and makes a formal case for the argument that the federal income tax schedule became vertically inequitable after the Tax Reform Act of 1986. In Section 5, we examine the technical requirements of our results in greater detail and point out a number of open problems which arise naturally in our framework. Section 6 supplies the proofs of our theorems and we conclude with an Appendix that discusses the negligibility of the 'pathological' cases that occur in our theory.

2. PRELIMINARIES

2.1. *Tax Functions*. Let N be any positive integer and let $0 < b_1 < b_2 <, \dots, < b_{N-1}$. We shall consider (statutory) tax functions t: $\mathbf{R}_+ \to \mathbf{R}_+$ that can be

[11] This result is true as long as the preferences of the individuals are assumed to be the same, that is, when the agents are treated as if they were all alike. Although one might feel that this is too strong an assumption, we should mention that it underlies most of the economic theories related to income redistribution, including, for instance, the theories of optimal income taxation and income inequality evaluation.

expressed as

$$(3)\qquad t(x)=\begin{cases}\alpha_1 x, & \text{if } 0\le x\le b_1\\ \alpha_2 x+\theta_1, & \text{if } b_1<x\le b_2\\ \dots\dots, & \dots\dots\\ \alpha_N x+\theta_{N-1}, & \text{if } b_{N-1}<x\end{cases}$$

for some real numbers $\alpha_1,\dots,\alpha_N$ with $\alpha_i\neq\alpha_{i+1}$, $i=1,2,\dots,N$, and $\theta_1,\dots,\theta_{N-1}$ such that

$$(4)\qquad 0<\alpha_i<1,\quad i=1,2,\dots,N$$

and

$$(5)\qquad \theta_i=t(b_i)-\alpha_{i+1}b_i,\quad i=1,2,\dots,N-1.$$

Associated with a tax function $t(\cdot)$ given by (3), (4) and (5), there is a post-tax function f: $\mathbf{R}_+\to\mathbf{R}_+$ defined as $f(x)=x-t(x)$ for all $x\ge 0$.

Let us denote the class of all tax functions given by (3), (4) and (5) by $\mathscr{T}^*(b_1,\dots,b_{N-1})$, and define

$$\mathscr{T}(N):=\bigcup_{0<b_1<\cdots<b_{N-1}}\mathscr{T}^*(b_1,\dots,b_{N-1}).$$

We interpret $\mathscr{T}(N)$ as the class of all tax functions polygonally defined on N many tax brackets. We therefore define the *class of all admissible* (*statutory*) *tax functions* as

$$\mathscr{T}:=\bigcup_{N\in\mathbf{Z}_{++}\setminus\{1\}}\mathscr{T}(N).$$

A number of observations about $\mathscr{T}$ is in order. First notice that any member $t(\cdot)$ of $\mathscr{T}$ is nonflat and defined on at least two tax brackets. Second, for any tax function, we have

$$t(0)=0\quad\text{and}\quad 0<t(x)<x \text{ for all } x>0.$$[12]

Finally, note that, by (5), any member of $\mathscr{T}$ is continuous, and by (4), it is strictly increasing and strongly incentive-preserving (cf. Fei 1981, and Eichhorn et al. 1984)

[12] Notice that negative income taxation is excluded from the analysis. This is because such a tax function does not impose any sacrifice on some members of the society, and hence by definition, cannot impose equal sacrifice upon all. Consequently, our entire analysis is conducted in terms of statutory income taxation, and therefore admittedly neglects many interesting dimensions of the actual income taxation practice (like tax deductions). Indeed, it is our intention to extend the analysis to the realm of *effective* income taxation in our future research. Nevertheless, we should mention that a referee of this journal argued that "the problem here is deeper.... The issue is that the equal sacrifice tax principle is basically a cost sharing approach to tax design. Negative income taxes are the result of using taxes as a way to affect the distribution of income; a completely different philosophy."

in the sense of guaranteeing that the ranking of taxpayers by pre-tax and post-tax income is the same. We should also mention that all personal income taxes of OECD countries (except Germany) do belong to $\mathscr{T}$. For example, the 1988 federal income tax schedule of the U.S.A. is a member of $\mathscr{T}(4)$ (cf. Pechmann 1987, p. 69).[13]

A tax function $t(\cdot)$ is said to be *progressive* (*regressive*) if $x \mapsto [t(x)]/x$ is a nondecreasing (nonincreasing) mapping on $\mathbf{R}_{++}$. We shall denote the class of all $t \in \mathscr{T}$ which are progressive (regressive) by $\mathscr{T}^{\text{prog}}$ ($\mathscr{T}^{\text{reg}}$, resp.).[14] An alternative progressivity concept is *marginal rate progressivity* which demands that the marginal tax rate is nondecreasing everywhere; that is, that the tax function is convex. We shall denote the set of all marginal rate progressive tax functions by $\mathscr{T}^{\text{conv}}$. Marginal rate progressivity is, of course, stronger than progressivity: $\mathscr{T}^{\text{conv}} \subset \mathscr{T}^{\text{prog}}$. We conclude this subsection by illustrating that this containment is indeed proper.

EXAMPLE 2.1. Define $t \in \mathscr{T}(3)$ as

$$t(x) = \begin{cases} x/4, & \text{if } 0 \le x \le 1 \\ (3x/4) - (1/2), & \text{if } 1 < x \le 2 \\ x/2, & \text{if } 2 < x. \end{cases}$$

One can easily see that while $x \mapsto t(x)/x$ defines an everywhere increasing mapping, $t(\cdot)$ is not convex around 2, and thus, $t \in \mathscr{T}^{\text{prog}} \setminus \mathscr{T}^{\text{conv}}$. □

2.2. *Utility Functions for Income.* A function $u: \mathbf{R}_{++} \to \mathbf{R}$ will be called a *utility function* for income if it is continuous and strictly increasing everywhere. The class of all utility functions for income is denoted by $\mathscr{U}$. One of the subclasses of $\mathscr{U}$ we shall be working with is

$$\mathscr{U}^{\bullet} := \{u \in \mathscr{U} : u \text{ is differentiable near origin}\}.$$

For $u \in \mathscr{U}^{\bullet}$, there must exist an interval $I = (0, a)$ with $a > 0$ (however small) such that u is differentiable on I. $\mathscr{U}^{\bullet}$ is clearly dense in $\mathscr{U}$ and it appears that it is only a minimal refinement of $\mathscr{U}$. In other words, it seems quite difficult to argue that one misses useful utility functions for income by concentrating on $\mathscr{U}^{\bullet}$ rather than $\mathscr{U}$.

A very important subclass of $\mathscr{U}^{\bullet}$ is

$$\mathscr{U}_0 := \{u \in \mathscr{U}^{\bullet} : u \text{ is concave on } \mathbf{R}_{++}\}.$$

[13] Of course, the fact that the real-world income tax functions are polygonal may not be viewed as sufficient motivation for restricting attention to piecewise linear tax functions. After all, such tax functions may simply be thought of as an administratively convenient approximation of the underlying *smooth* tax functions. Having said this, however, we should note that the present exercise gets technically complicated when one allows for smooth tax functions. We have indeed taken on such an analysis elsewhere in terms of right differentiable tax functions (see Mitra and Ok 1995), and have shown that the basic argument we develop in the present paper remains valid with such tax functions.

[14] Notice that if the tax function given by (3), (4) and (5) is progressive, then $t(x)/x$ must be strictly increasing on at least (b_1, b_2) since $\alpha_1 \neq \alpha_2$.

$\mathscr{U}_0$ will be referred to as the *class of all admissible utility functions*. By virtue of the standard arguments favoring risk-averse behavior, we believe restricting attention to concave utility functions is justified. Moreover, the declining marginal utility of income guarantees that the tax liability of an equal sacrifice tax is increasing in income. Many classical writers believed that this property is also sufficient to warrant the progressivity of an equal sacrifice tax function. We shall later show that, in the domain of polygonal tax functions, this perception (although not shared by many contemporary economists) was correct. But, this is going ahead of our story.

2.3. *Equal Sacrifice Tax Functions*. Consider the following statement:

$$\exists c > 0 : [\forall x > 0 : u(x) - u(x - t(x)) = c] \tag{6}$$

where $u : \mathbf{R}_{++} \to \mathbf{R}$ is any function. One may consider qualifying a tax function as equal sacrifice whenever (6) holds for at least one $u \in \mathscr{U}$. This is, in fact, precisely how Young (1988) defines equal sacrifice taxes. But with only the restrictions of monotonicity and continuity on the utility functions, we cannot get strong implications from the theory since, with Young's definition, *any* tax function is an equal sacrifice tax:

THEOREM 2.2. *For any given $t \in \mathscr{T}$ and any sacrifice level $c > 0$, there exists a utility function $u \in \mathscr{U}$ such that $u(x) - u(x - t(x)) = c$ for all $x > 0$.*

(A stronger version of this result is established in Ok (1995).) In fact, this result remains valid even if we restrict the class of utility functions to $\mathscr{U}^{\bullet}$:

THEOREM 2.3. *For any given $t \in \mathscr{T}$ and any sacrifice level $c > 0$; there exists a utility function $u \in \mathscr{U}^{\bullet}$ such that $u(x) - u(x - t(x)) = c$ for all $x > 0$.*

(A proof of this theorem is given in Section 6.) The criticism of Samuelson's reasoning noted in the introduction is solely based on the possibility that, given a tax function $t(\cdot)$, (1) cannot be satisfied for a large class of utility functions. Theorem 2.3 shows that, for any given $t \in \mathscr{T}$, the functional equation (1) has certainly a solution in $\mathscr{U}^{\bullet}$, and, therefore, demanding the utility function to be differentiable near zero does not accomplish anything beyond Samuelson's analysis.[15] To learn something new, we have to put another restriction on the class of utility functions. A rather natural restriction is, of course, *concavity*.

We have already mentioned that demanding the concavity of the utility functions of the representative agents is essential to the theory. Indeed, the assumption of decreasing marginal utility is almost exclusively made in the related literature. We shall therefore say that a tax function $t(\cdot)$ is an *equal sacrifice tax* if (6) holds for some $u \in \mathscr{U}_0$, that is, the *set of all equal sacrifice tax functions* is defined as

$$\mathscr{A} := \{t \in \mathscr{T} : (6) \text{ holds for some } u \in \mathscr{U}_0\}.$$

[15] Theorem 2.3 also clarifies that requiring $u(\cdot)$ to be differentiable near origin while keeping $t(\cdot)$ not differentiable at finitely many points does not pose any problem at all with regard to satisfying (1) everywhere.

REMARK 2.4. (i) The above statements concentrate only on equal *absolute* sacrifice. One might also want to study equal *proportional* sacrifice by replacing (6) with the statement

$$\exists c > 0 : [\forall x > 0 : u(x) = cu(x - t(x))].$$

This is, however, redundant for it is easy to see that $\mathscr{A} := \{t \in \mathscr{T} : (7) \text{ holds for some } u \in \mathscr{U}_0\}$.

(ii) The way we define equal sacrifice is best interpreted by considering $u(\cdot)$ as standing for the preferences of a *representative agent* of the society, and thereby acting as a *social norm* (cf. Musgrave 1959, and Young 1990), or simply as representing the preferences of the tax planner for income (cf. Stern 1977). □

We conclude this section by stressing the importance of the set $\mathscr{T} \setminus \mathscr{A}$. Since a particular $u \in \mathscr{U}_0$ relative to which $t \in \mathscr{T}$ satisfies (6) may not be the correct social norm, we cannot be sure that it is perfectly *vertically equitable* simply because $t \in \mathscr{A}$. On the other hand, if $t \notin \mathscr{A}$, then we can infer that t cannot inflict the same sacrifice upon all relative to *any* admissible social norm. One may then justly view $\mathscr{T} \setminus \mathscr{A}$ as the set of all *vertically inequitable* taxes. In the next section we will show that almost any tax function which is not marginal-rate progressive can be viewed as vertically inequitable.

3. TAX PROGRESSIVITY AND THE PRINCIPLE OF EQUAL SACRIFICE

3.1. *Marginal Rate Progressivity and the Principle of Equal Sacrifice.* We start our analysis by identifying a fundamental subclass of equal-sacrifice tax functions. In particular, our first result states that any convex tax schedule is, in fact, an equal-sacrifice tax function. The essence of the theorem is given in the following lemma.

LEMMA 3.1.[16] *Let $t \in \mathscr{T}$, $f(x) = x - t(x)$ for all $x \geq 0$ and $c > 0$. If there exist an integer M and an increasing and unbounded sequence $\{\varphi_n\}_{n=1}^{\infty}$ such that, for any $n \geq M$, $f^n(\cdot)$ is concave on $[0, \varphi_n]$, then there exists an admissible utility function $u \in \mathscr{U}_0$ such that*

$$u(x) - u(f(x)) = c \text{ for all } x > 0.$$

We note the simple fact that if $t \in \mathscr{T}^{\text{conv}}$, then the post-tax function $f(\cdot)$ (defined as $f(x) := x - t(x)$ for all $x \geq 0$) is concave, which, in turn, implies that, for any

[16] For any $g\colon A \to \mathbf{R}$, $A \subseteq \mathbf{R}$, and positive integer k, we define the kth *iterate* of g (denoted by $g^k(\cdot)$) as

$$g^k(\cdot) := (g \circ g \circ \cdots \circ g)(\cdot)$$

where the composition operator is applied k times.

$n \geq 1$, $f^n(\cdot)$ is concave everywhere. Therefore, by Lemma 3.1, we have

THEOREM 3.2. *If $t \in \mathcal{T}^{\text{conv}}$, that is, if $t(\cdot)$ is a marginal-rate progressive tax, then there exist a sacrifice level $c > 0$ and an admissible utility function $u \in \mathcal{U}_0$ such that*

$$u(x) - u(x - t(x)) = c \text{ for all } x > 0.$$

In short, $\mathcal{T}^{\text{conv}} \subset \mathcal{A}$.

There are nonconvex tax functions for which (6) holds. However, we shall see later that these tax functions can be regarded as '*pathological*' cases.

EXAMPLE 3.3. Consider the tax function introduced in Example 2.1. The post-tax function implied by this tax schedule can be written as

$$f(x) = \begin{cases} 3x/4, & \text{if } 0 \leq x \leq 1 \\ (x/4) + (1/2), & \text{if } 1 < x \leq 2. \\ x/2, & \text{if } 2 < x \end{cases}$$

Let $n \geq 3$. Then,

$$f^n(x) = \begin{cases} \left(\frac{3}{4}\right)^n x, & \text{if } 0 \leq x \leq 1 \\ \left(\frac{3}{4}\right)^{n-1}\left(\frac{1}{4}x + \frac{1}{2}\right), & \text{if } 1 < x \leq 2 \\ \left(\frac{3}{4}\right)^{n-2}\left(\frac{1}{4}\left(\frac{x}{2}\right) + \frac{1}{2}\right), & \text{if } 2 < x \leq f^{-1}(2) \\ \dots, & \dots \\ \left(\frac{3}{4}\right)\left(\frac{1}{4}\left(\frac{x}{2^{n-2}}\right) + \frac{1}{2}\right), & \text{if } f^{2-n}(2) < x \leq f^{1-n}(2) \\ \frac{1}{4}\left(\frac{x}{2^{n-1}}\right) + \frac{1}{2}, & \text{if } f^{1-n}(2) < x \leq f^{-n}(2) \\ \frac{x}{2^n}, & \text{otherwise.} \end{cases}$$

It is easy to observe that $f^n(\cdot)$ is concave on $[0, f^{-n}(2)]$ and that $\{f^{-n}(2)\}_{n=1}^{\infty}$ is a monotonic sequence such that $\lim_{n\to\infty} f^{-n}(2) = \lim_{n\to\infty} 2^{n+1} = \infty$. Therefore, we can apply Lemma 3.1 and conclude that there exists a $(c, u) \in \mathbf{R}_{++} \times \mathcal{U}_0$ such that $u(x) - u(x - t(x)) = c$ for all $x > 0$; that is, $t(\cdot)$ is an equal sacrifice tax. □

REMARK 3.4. The peculiarity of the tax function studied in Example 3.3 is that its post-tax function has the property $f(2) = 1$; that is, the post-tax income of the highest income earner in the second tax bracket is exactly equal to the highest income in the first tax bracket. We can, in fact, generalize the idea behind this example. Let $t(\cdot)$ be tax function defined on N many tax brackets and let $t \in \mathscr{T}^{\text{prog}} \setminus \mathscr{T}^{\text{conv}}$. If

$$b_i - t(b_i) = b_{i-1} \text{ for all } i = 2, 3, \ldots, N-1,$$

then $t(\cdot)$ must be an equal sacrifice tax. The proof is omitted (but is, of course, available upon request) since it is rather cumbersome and analogous to the one given in Example 3.3 in spirit. □

3.2. *Vertically Non-equitable Taxation Schemes.* Our main purpose in this subsection is to argue that any given nonconvex tax function which satisfies a certain regularity condition is, in fact, inequitable from the point of view of the ability to pay doctrine. We shall do this by showing that such a tax function cannot be justified as an equal sacrifice tax with respect to *any* admissible utility function. Our main result (the proof of which is relegated to Section 5) takes the following form:

THEOREM 3.5. *Let* $N \geq 2$, $0 < b_1 < b_2 < \cdots < b_{N-1}$, *and* $t(\cdot)$ *be a tax function that can be written as in* (3), (4) *and* (5). *Assume that* $t(\cdot)$ *is not convex*; *that is,*

$$\alpha_{j+1} < \alpha_j \textit{ for some } j \in \{1, 2, \ldots, N\},$$

and denote the smallest such j *by* j_0. *Then, if, for every positive integer* n *and every* $k \in \{1, 2, \ldots, j_0 - 1\}$,

$$f^n(b_{j_0}) \neq b_k,$$

there does not exist $(c, u) \in \mathbf{R}_{++} \times \mathscr{U}_0$ *such that*

$$u(x) - u(x - t(x)) = c \textit{ for all } x > 0.$$

Notice that if $t \in \mathscr{T}^{\text{reg}}$, then $\alpha_2 < \alpha_1$ (i.e., $j_0 = 1$), and therefore, all the requirements of Theorem 3.5 are trivially met. Consequently, contrary to the standard literature in public finance, we may conclude that an equal sacrifice tax *cannot* be regressive. In other words,

COROLLARY 3.6. *A regressive tax function cannot impose equal sacrifice upon everyone for any utility function* $u \in \mathscr{U}_0$. *In short,* $\mathscr{T}^{\text{reg}} \cap \mathscr{A} = \varnothing$.

In stating Theorem 3.5 and Corollary 3.6, we did not assume any additional restriction on the behavior of the marginal utility schedules beyond those implied by the fact that the utility function is in the set $\mathscr{U}_0$. Therefore, the well-known claim that a regressive tax function may equate sacrifices with respect to a concave utility function with the elasticity of marginal utility being less than unity (see, among many others, Musgrave and Musgrave 1980, p. 251) has to be rejected when one confines attention to piecewise linear tax functions.[17]

[17] Corollary 3.6 shows that, given a tax function $t \in \mathscr{T}^{\text{reg}}$, (1) cannot be satisfied for *any* concave utility function which is differentiable near origin. Whether one can establish this result without restricting the utility functions to be differentiable near origin or not, is of course a question of interest. This question is open at the moment.

REMARK 3.7. Theorem 3.5 is by no means a trivial consequence of the simple incompatibility between the differentiability of $u(\cdot)$ near the origin and the nondifferentiability of $t(\cdot)$ at finitely many points. (Indeed, see Theorem 2.3 and footnote 15). Since it seems extremely difficult to argue that individuals' utility functions for income are concave and strictly increasing functions which are not differentiable in *any* neighborhood of the origin, we believe that Corollary 3.6 illustrates that there is a genuine incompatibility between regressive taxes and the principle of equal sacrifice.

3.3. *An Equal Sacrifice Characterization of Marginal Rate Progressivity.* We now refine the class of all admissible tax functions $\mathscr{T}$ by eliminating certain tax schedules which we view as 'pathological'. Let N be a positive integer, $0 < b_1 < b_2 < \cdots < b_{N-1}$, and define $\mathscr{T}_0^*(b_1, \ldots, b_{N-1})$ as the class of all members of $\mathscr{T}^*(b_1, \ldots, b_{N-1})$ (the set of all tax schedules defined on the tax brackets $[0, b_1], [b_1, b_2], \ldots, [b_{N-1}, \infty)$) such that the induced post-tax functions have the following technical property: for every positive integer n, every $k \in \{1, 2, \ldots, i-1\}$ and every $i \in \{2, 3, \ldots, N-1\}$,

$$f^n(b_i) \neq b_k.$$

We then modify the admissibility of a tax function by refining $\mathscr{T}$ to obtain

$$\mathscr{T}_0 := \bigcup_{N \in \mathbf{Z}_{++} \setminus \{1\}} \bigcup_{0 < b_1 < b_2 < \cdots < b_{N-1}} \mathscr{T}_0^*(b_1, \ldots, b_{N-1}).$$

Now, we argue that restricting attention to $\mathscr{T}_0$ (as opposed to $\mathscr{T}$) comes without a significant loss in generality. One argument comes readily from a practical angle. Of the 17 OECD countries reported in OECD (1986), none has applied a personal income tax function that belongs to $\mathscr{T} \setminus \mathscr{T}_0$ in the fiscal time period 1975–1984. We can, in fact, justify this occurrence from a measure theoretical perspective. In the Appendix, for any given $N \geq 2$ we shall define a measure on $\mathscr{T}(N)$ in a very natural way and show that $\mathscr{T}(N) \setminus \mathscr{T}_0$ is in fact a set of measure zero. It is for this reason we qualify the tax functions that belong to $\mathscr{T} \setminus \mathscr{T}_0$ as 'pathological'.[18]

Once we restrict attention to $\mathscr{T}_0$ (and accept that this is essentially without loss of generality), the argument given in Theorem 3.5 becomes quite strong:

COROLLARY 3.8. *A tax function in $\mathscr{T}_0 \setminus \mathscr{T}^{\text{conv}}$ cannot impose equal sacrifice upon everyone for any utility function $u \in \mathscr{U}_0$. In short, $\mathscr{A} \cap (\mathscr{T}_0 \setminus \mathscr{T}^{\text{conv}}) = \varnothing$.*

This result implies that even (average rate) progressivity might not be enough to ensure that equal sacrifice principle holds with respect to at least one admissible utility function. Here is a concrete example.

[18] One can also show that $\mathscr{T} \setminus \mathscr{T}_0$ is, in fact, a nowhere dense set in $\mathscr{T}$.

EXAMPLE 3.9. Consider the following tax function

$$t(x) = \begin{cases} x/4, & \text{if } 0 \le x \le 1 \\ (x/2) - (1/4), & \text{if } 1 < x \le 2 \\ 3x/8, & \text{if } 2 < x. \end{cases}$$

Notice that $t(\cdot)$ is progressive but not convex. Moreover, $t \in \mathscr{T}_0$ since $2 - t(2) = \frac{5}{4} \neq 1$ and $\frac{5}{4} - t(\frac{5}{4}) = \frac{7}{8} \neq 1$. Therefore, in view of Corollary 3.8, $t(\cdot)$ is not an equal sacrifice tax function even though it is progressive. □

We conclude this subsection by noting that Theorem 3.2 (which states that any convex tax schedule is an equal sacrifice tax) and Corollary 3.8 together yield an equal sacrifice characterization of marginal rate progressive tax functions that belong to $\mathscr{T}_0$:

COROLLARY 3.10. *The tax function $t(\cdot)$ belongs to $\mathscr{T}^{\text{conv}} \cap \mathscr{T}_0$ if, and only if, there exists $(c, u) \in \mathbf{R}_{++} \times \mathscr{U}_0$ such that*

$$u(x) - u(x - t(x)) = c \text{ for all } x > 0.$$

In short, $\mathscr{T}^{\text{conv}} \cap \mathscr{T}_0 = \mathscr{A} \cap \mathscr{T}_0$.

REMARK 3.11. It is possible to make use of the principle of equal sacrifice to uncover the government's valuation of income (cf. Stern 1977, and Young 1990). The question then is, as Stern puts it, of the inverse optimum variety: assuming that the government chooses a (statutory) income tax schedule according to the principle of equal sacrifice, given an observed (statutory) income tax scheme, what can we infer about the preferences of the government for income? It is important to note that Corollary 3.10 provides the following insight with respect to this question. Observing a statutory income tax function (in $\mathscr{T}_0$) which is *not* marginal rate progressive, either the government's preferences for income *cannot* be represented by a concave utility function, or the government simply does not adhere to the principle of equal sacrifice.

3.4. *Relation to Inequality-Averse Taxation.* It is well known that a tax function reduces income inequality (in the sense of shifting the Lorenz curve of the pre-tax distribution upward) for any given pre-tax income distribution if, and only if, it is progressive.[19] Due to this fact, inequality-averse taxation is typically identified with progressive taxation. Therefore, by Corollary 3.10, we establish the fact that equal sacrifice taxation is inequality-reducing:

COROLLARY 3.12. *Let $t(\cdot)$ be an equal-sacrifice tax in $\mathscr{T}_0$. Then, for any $\mathbf{x} \in \mathbf{R}^n_+$, $n \ge 2$, the post-tax distribution $\mathbf{x} - (t(x_1), t(x_2), \ldots, t(x_n))$ Lorenz dominates $\mathbf{x}$.*

[19] This result is usually attributed to Jakobsson (1976), Fellman (1976) and Kakwani (1977). The stated version of the theorem is proved in Eichhorn et al. (1984). The most general formulations of the theorem appear in Thon (1987) and Le Breton et al. (1996). For local formulations of the result, we refer the reader to Hemming and Keen (1983) and Latham (1988).

Again by Corollary 3.10, however, the converse of this proposition is not true, for, as Example 3.9 illustrates, a progressive tax schedule need not be an equal-sacrifice tax. Consequently, we learn that equal-sacrifice taxation is more demanding than (relative) inequality-reducing taxation. Since the principle of equal sacrifice, or more generally, the doctrine of ability to pay is typically viewed as a "fairness" criterion with regard to personal income taxation, this observation shows that the properties of fairness and income inequality aversion *cannot* always be viewed as identical, contrary to what seems to be the common belief in the related literature.

4. AN EQUAL SACRIFICE THEORETICAL EVALUATION OF THE 1986 TAX REFORM ACT

Slemrod (1990, p. 166) states that

> When the optimal progressivity literature first surfaced in the early 1970s the top marginal tax rate stood at 70 percent.... As of January 1, 1988, the marginal tax rate on the highest income has fallen to 28 percent, a remarkably steep drop.... a key message of the optimal progressivity literature, that high marginal rates may not be appropriate even for egalitarian social welfare functions, has apparently won the day.

Slemrod's point is well supported by the Tax Reform Act of 1986, undoubtedly the most comprehensive tax reform in the history of U.S. federal income taxation. Indeed, one of the properties of this tax reform that was subject to an intense scrutiny, was its sharp reduction in upper bracket tax rates which meant a clear withdrawal from progressive taxation.[20] This may easily be observed by comparing Tables 1 and 2 which, respectively, reflect the U.S. federal personal income tax schedules in 1987 and 1988. (The reader will recall that the tax rates enacted by the 1986 tax reform became fully effective in 1988.) In what follows, we shall compare these tax functions drawing upon our previous results.

An important thing to notice in Table 1 is that the 1987 tax schedule was a marginal rate progressive tax function. By virtue of Theorem 3.2, we can therefore conclude that the 1987 tax function is an equal-sacrifice tax. (We should emphasize that this finding applies to all federal individual income tax functions that are used in the fiscal period 1975–1987.)[21]

[20] However, due to the accompanying broadening of the tax base, the effective tax rates were left largely unaffected (see, for instance, Pechmann 1990, and Kasten et al. 1994). In fact, Pechmann (1990), p. 12, argues that "the distribution effect of the 1986 act is distinctly progressive, especially if the increase in corporate tax liabilities is taken into account." Nevertheless, we should stress that Pechmann (along with most public finance specialists) identifies the *fairness* of a tax system with its *inequality-averse behavior*. The remarks in Section 3.4 following Corollary 3.12, on the other hand, make clear that such an approach might be inappropriate.

[21] Of course, this statement is meaningful only with respect to our definition of equal-sacrifice taxation. All Theorem 3.2 tells us is that there exists a well-behaved utility function relative to which the 1987 tax schedule would inflict equal sacrifice upon all individuals. Without knowing whether this utility function is a good representative of the true preferences of the individuals or not, we cannot simply say that the 1987 tax schedule is vertically equitable.

TABLE 1

1987 Federal Personal Income Tax Rates*

Taxable Income			
Married persons filing joint returns	Married persons filing separate returns	Single persons	Tax Rates
0–3,000	0–1,500	0–1,800	11%
3,000–28,000	1,500–14,000	1,800–16,800	15%
28,000–45,000	14,000–22,500	16,800–27,000	28%
45,000–90,000	22,500–45,000	27,000–54,000	35%
Over 90,000	Over 45,000	Over 54,000	38.5%

* Source: Pechmann (1987), Table 4-2, p. 69.

TABLE 2

1988 Federal Personal Income Tax Rates*

Taxable Income			
Married persons filing joint returns	Married persons filing separate returns	Single persons	Tax Rates
0–29,750	0–14,875	0–17,850	15%
29,750–71,900	14,875–35,950	17,850–43,150	28%
71,900–149,250	35,950–113,300	43,150–89,560	33%
Over 149,250	Over 113,300	Over 89,560	28%

* Source: Pechmann (1987), Table 4-2, p. 69.

Turning our attention to Table 2, we see quite a different picture. First, notice that the 1988 federal personal income tax schedule is a progressive tax which is *not* convex. Second, one can easily check that it is a member of $\mathscr{T}_0$; that is, it is not a 'pathological' case. Therefore, we can apply Corollary 3.10 and conclude that the 1988 statutory income tax function is certainly an inequitable tax schedule with respect to the principle of equal sacrifice. Put precisely, there does not exist a single admissible social norm relative to which the 1988 statutory income tax schedule imposes the same level of sacrifice upon everyone.[22,23]

We should conclude by warning the reader that the above evaluation of the Tax Reform Act of 1986 is utterly incomplete. All we checked above is the sacrifice equitability of the federal personal income taxation before and after the reform. Our analysis clearly falls short of evaluating inequitability of the effective tax functions induced by the 1986 Tax Reform Act, and indeed this is hardly a minor shortcoming. Moreover, the 1986 reform has undoubtedly brought out many major changes apart from envisaging a significant reduction in the upper bracket tax rates.

[22] This observation seems in line with Young (1990) who showed that a particular equal-sacrifice model fits U.S. tax schedules in the post-war period until the Tax Reform Act of 1986 was enacted.

[23] This conclusion holds true only in the period of 1988–1991. Starting from 1991, the U.S. federal personal income tax schedule became once again marginal-rate progressive.

Our present focus is admittedly limited and cannot be expected to yield extensive insights about the equity-related consequences of the reform in general.[24] We are, therefore, tempted to view the analysis of this section as nothing but an illustration of the (potential) practical implications of our theoretical work.

5. OPEN PROBLEMS

Remark 3.7 notes that Theorem 3.5 is not a trivial consequence of a simple incompatibility between differentiability of an admissible utility function near zero and nondifferentiability of an admissible tax function (at finitely many points). Indeed, Theorem 2.3 vigorously justifies this claim. However, we should mention that the proof of Theorem 3.5 does exploit these properties of utility and tax functions, and, therefore, we should discuss the implications of dropping the differentiability assumption of the utility functions. The problem is then to determine the set of all tax functions in $\mathscr{T}$ (call it $\mathscr{B}$) such that there is not a strictly increasing and concave $u: \mathbf{R}_+ \to \mathbf{R}_+$, with

$$u(x) - u(x - t(x)) = \text{constant for all } x > 0.$$

This problem remains open at the moment.

From the perspective of the doctrine of ability to pay, $\mathscr{B}$ is composed of inequitable taxes in a very strong sense. A question to ask is whether there exists any progressive taxes in $\mathscr{B}$ at all. If the answer is yes (and it is), then this means that the principle of equal sacrifice can be effectively used in assessing the normative properties of progressive taxes.

This appears to be a natural way of making use of the principle of equal sacrifice (see Mitra and Ok 1995). It seems to us that the reason why this question is not at all addressed in the literature is because Samuelson's analysis is usually taken to imply that the principle of equal sacrifice has no selective power. Many authors appear to indicate that any tax function can be equal sacrifice with respect to *some* utility function (i.e., that $\mathscr{B} = \varnothing$). As noted earlier, this conclusion not only does not follow from Samuelson's demonstration but also is incorrect. In fact, the following result shows that $\mathrm{B} \cap \mathscr{T}^{\text{prog}} \neq \varnothing$.

PROPOSITION 5.1. *Let*

$$t(x) = \begin{cases} \alpha_1 x, & x \in [0, b_1] \\ \alpha_2 x + \theta_1, & x \in (b_1, b_2] \\ \alpha_3 x + \theta_2, & x \in (b_2, \infty) \end{cases}$$

and assume that $t \in \mathscr{T}(3)$. *If*

$$(1 - \alpha_3) b_2 - \theta_2 < b_1 \tag{8}$$

[24] For extensive evaluations of the 1986 Tax Reform Act, we refer the reader to Pechmann (1987, 1990), Slemrod (1990) and McLure and Zodrow (1994), among others.

and

$$(9) \qquad \frac{1-\alpha_3}{1-\alpha_2} > \frac{1}{1-\alpha_1},$$

then there does not exist a strictly increasing and concave utility function such that $u(x) - u(x - t(x))$ *is constant for all* $x > 0$.[25]

REMARK 5.2. (i) Since the restrictions (8) and (9) are in terms of *strict* inequalities, the proposition is robust.

(ii) This proposition establishes that Theorem 2 of Ok (1995) cannot be extended to the class of all progressive tax functions, and hence, answers the open question stated therein in the negative.

(iii) One can also show similarly that $B \cap \mathscr{T}^{\text{reg}} \neq \varnothing$. □

By virtue of Proposition 5.1, we conclude that the principle of equal sacrifice does have a 'bite' in assessing the equity properties of progressive taxes. Determination of $\mathscr{B} \cap \mathscr{T}^{\text{prog}}$ is then of interest, for only then one will know which progressive taxes are, in fact, vertically inequitable. This question is also open and will be a topic of future research.

6. PROOFS

6.1. *Proof of Theorem* 2.3. Fix a positive integer N and let $0 < b_1 < b_2 < \cdots < b_{N-1}$. Assume that $t \in \mathscr{T}^*(b_1, \dots, b_{N-1})$ and that all the hypotheses of the lemma hold. Let, for any $x > 0$,

$$G_n(x) := \frac{f^n(x)}{\lambda^n} \text{ for all } n = 1, 2, \dots$$

where $\lambda := f'(0) = 1 - t'(0)$ and $f^n(\cdot) = f(f(\dots(f(\cdot)))$, and define the mapping G: $\mathbf{R}_{++} \to \mathbf{R}_{++}$ as

$$G(x) := \lim_{n \to \infty} G_n(x).$$

To see that $G(\cdot)$ is well-defined, pick any $y > 0$ and define the sequence $\{y_n\}_{n=0}^{\infty}$ as

$$y_n := f^n(y), \quad n = 0, 1, \dots .$$

Since $0 < f(x) < x$ for all $x > 0$, $\{y_n\}_{n=0}^{\infty}$ is monotonically decreasing and is bounded below. Therefore, there exists $\hat{y} \geq 0$ such that $\lim_{n \to \infty} y_n = \hat{y}$. If $\hat{y} > 0$, then by the continuity of $f(\cdot)$,

$$f(\hat{y}) = f\Big(\lim_{n \to \infty} y_n\Big) = \lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} y_{n+1} = \hat{y},$$

[25] If $\alpha_2 > \alpha_1$ and $\alpha_2 b_2 + \theta_1 < \alpha_3 b_2$, then this tax function is necessarily progressive. For a numerical example satisfying all these conditions, take $(\alpha_1, \alpha_2, \alpha_3) = (0.1, 0.4, 0.3)$, $(b_1, b_2) = (80, 90)$, and $(\theta_1, \theta_2) = (-24, -15)$.

and this contradicts the fact that the only fixed point of $f(\cdot)$ is zero. Thus, $\hat{y}=0$ and this means that there is a positive integer K such that $y_n \in [0, b_1]$ for all $n \geq K$. Hence, for $n \geq K+1$,

$$y_n = f(y_{n-1}) = \lambda y_{n-1},$$

that is, $(y_n/y_{n-1}) = \lambda$ for all $n = K+1, K+2, \dots$. Therefore, for all $n \geq K+1$,

$$\frac{f^n(y)}{\lambda^n} = \frac{y_0 \Pi_{i=1}^n (y_i/y_{i-1})}{\lambda^n} = \frac{y_0 \Pi_{i=1}^K (y_i/y_{i-1})}{\lambda^K} = \frac{f^K(y)}{\lambda^K}.$$

Consequently,

$$\lim_{n\to\infty} G_n(y) = \lim_{n\to\infty} \frac{f^n(y)}{\lambda^n} = \frac{f^K(y)}{\lambda^K}$$

and since y was arbitrary in this analysis, we conclude that $G(\cdot)$ is well-defined.

We now show that $G(\cdot)$ is continuous and strictly increasing. Pick again an arbitrary $y>0$ and $z>y$. By the above procedure, one can find a positive integer L such that, for any $n \geq L+1$,

$$\frac{f^n(z)}{\lambda^n} = \frac{f^L(z)}{\lambda^L}.$$

It then follows that

$$\frac{f^n(x)}{\lambda^n} = \frac{f^L(x)}{\lambda^L} \quad \text{for all } x \in (0, z]$$

so that

$$G(x) = \frac{f^L(x)}{\lambda^L} \quad \text{for all } x \in (0, z].$$

It now follows readily from the continuity and strict monotonicity of $f(\cdot)$ that $G(\cdot)$ is continuous at y and that $G(z) > G(y)$. Since $y>0$ and $z>y$ were arbitrary, we conclude that $G(\cdot)$ must be continuous and strictly increasing on $\mathbf{R}_{++}$.

Now, pick any $c>0$, and define $u\colon \mathbf{R}_{++} \to \mathbf{R}$ as

$$u(x) = \frac{-c}{\log \lambda} \log G(x) \text{ for all } x>0.$$

By the observations above, $u(\cdot)$ is a continuous and strictly increasing function on $\mathbf{R}_{++}$. Furthermore, since $G(x)=x$ for all $x \in (0, b_1]$, $u(\cdot)$ is differentiable near origin. Therefore, $u \in \mathscr{U}^\bullet$ and for any $x > 0$,

$$\begin{aligned} u(x - t(x)) &= u(f(x)) \\ &= \frac{-c}{\log \lambda} \log G(f(x)) \\ &= \frac{-c}{\log \lambda} \log\left(\lim_{n\to\infty} \frac{f^{n+1}(x)}{\lambda^n} \right) \\ &= \frac{-c}{\log \lambda} \log\left(\lambda \lim_{n\to\infty} \frac{f^{n+1}(x)}{\lambda^{n+1}} \right) \\ &= -c + \frac{-c}{\log \lambda} \log\left(\lim_{n\to\infty} \frac{f^{n}(x)}{\lambda^n} \right) \\ &= \frac{-c}{\log \lambda} \log G(x) - c \\ &= u(x) - c. \end{aligned}$$

The proof of Theorem 2.3 is now complete.

6.2. *Proof of Lemma* 3.1. Define $G(\cdot)$ and $u(\cdot)$ as in the proof of Theorem 2.3. Then, $G(\cdot)$ is a concave function under the hypotheses of Lemma 3.1. Since $u(\cdot)$ is a concave and increasing transform of $G(\cdot)$, it is also concave.

6.3. *Proof of Theorem* 3.5. Seeking to derive a contradiction, assume the hypotheses of Theorem 3.5 (but for simplicity write j for j_0), and suppose that

$$u(x) - u(f(x)) = c \text{ for all } x > 0,$$

for some $(c, u) \in \mathbf{R}_{++} \times \mathscr{U}_0$. Then, for all $\epsilon > -b_j$, we must have

$$u(b_j + \epsilon) - u\big(f(b_j + \epsilon)\big) = c$$

and

$$u(b_j) - u\big(f(b_j)\big) = c$$

so that, for $\epsilon > 0$ small enough,

$$\begin{aligned} u(b_j + \epsilon) - u(b_j) &= u\big(f(b_j + \epsilon)\big) - u\big(f(b_j)\big) \\ &= u\big((1 - \alpha_{j+1})b_j - \theta_j + (1 - \alpha_{j+1})\epsilon\big) - u\big((1 - \alpha_{j+1})b_j - \theta_j\big), \end{aligned}$$

that is,

$$\frac{u(b_j+\epsilon)-u(b_j)}{\epsilon}$$
$$=(1-\alpha_{j+1})\left(\frac{u\big((1-\alpha_{j+1})b_j-\theta_j+(1-\alpha_{j+1})\epsilon\big)-u\big((1-\alpha_{j+1})b_j-\theta_j\big)}{(1-\alpha_{j+1})\epsilon}\right).$$

Letting $\epsilon \to 0+$, we obtain

$$u'_+(b_j)=(1-\alpha_{j+1})u'_+\big((1-\alpha_{j+1})b_j-\theta_j\big). \tag{10}$$

Similarly,

$$u'_-(b_j)=(1-\alpha_j)u'_-\big((1-\alpha_{j+1})b_j-\theta_j\big) \tag{11}$$

holds. By (10), (11) and the concavity of $u(\cdot)$,

$$u'_-\big((1-\alpha_{j+1})b_j-\theta_j\big)=\frac{1}{1-\alpha_j}u'_-(b_j)\geq\frac{1}{1-\alpha_j}u'_+(b_j)$$
$$=\frac{(1-\alpha_{j+1})}{(1-\alpha_j)}u'_+\big((1-\alpha_{j+1})b_j-\theta_j\big)$$

that is,

$$u'_-\big(f(b_j)\big)\geq\frac{(1-\alpha_{j+1})}{(1-\alpha_j)}u'_+\big(f(b_j)\big). \tag{12}$$

Now, let

$$m:=\min\{n\in\mathbf{Z}_{++}:f^n(b_j)<b_1\}.$$

By the hypothesis that $f^n(b_j)\neq b_k$ for any $k\in\{1,2,\dots,j-1\}$ and $n\in\mathbf{Z}_{++}$, we must have

$$f(b_j)\in(b_{\ell_1-1},b_{\ell_1}),\ \text{for some } \ell_1\in\{2,\dots,j\}$$
$$f^2(b_j)\in(b_{\ell_2-1},b_{\ell_2}),\ \text{for some } \ell_2\in\{2,\dots,\ell_1\}$$

. . . .

$$f^{m-1}(b_j)\in(b_{\ell_{m-1}-1},b_{\ell_{m-1}}),\ \text{for some } \ell_{m-1}\in\{2,\dots,\ell_{m-2}\}$$
$$f^m(b_j)\in(0,b_1).$$

Therefore, proceeding by the way we obtained (10), we have

$$(13)\quad \begin{aligned} u'_-\big(f^{n-1}(f(b_j))\big) &= (1-\alpha_{\ell_n})u'_-\big(f^n(f(b_j))\big), \quad n=1,2,\dots,m-1\\ u'_+\big(f^{n-1}(f(b_j))\big) &= (1-\alpha_{\ell_n})u'_+\big(f^n(f(b_j))\big), \quad n=1,2,\dots,m-1 \end{aligned}$$

and

$$(14)\quad \begin{aligned} u'_-\big(f^{n-1}(f(b_j))\big) &= (1-\alpha_1)u'_-\big(f^n(f(b_j))\big), \quad n=m,m+1,\dots\\ u'_+\big(f^{n-1}(f(b_j))\big) &= (1-\alpha_1)u'_+\big(f^n(f(b_j))\big), \quad n=m,m+1,\dots. \end{aligned}$$

Since $u(\cdot)$ is differentiable near origin, there exists $0<\gamma<b_1$ such that $u'(\cdot)$ is well-defined on $(0,\gamma)$. But, since $\lim_{n\to\infty} f^n(b_j)=0$ (as established in the previous subsection), there exists a positive integer $\hat{m}$ such that

$$(15)\qquad 0<f^n(b_j)<\gamma, \quad n=\hat{m},\hat{m}+1,\dots.$$

So, defining $M := \max\{m,\hat{m}\}$, by (12), (13) and (14),

$$\begin{aligned} u'_-\big(f^M(f(b_j))\big) &= \frac{1}{(1-\alpha_{\ell_1})\cdot(1-\alpha_{\ell_{m-1}})}\cdot\frac{u'_-\big(f(b_j)\big)}{(1-\alpha_1)^{M-m+1}}\\ &\geq \left(\frac{1}{(1-\alpha_{\ell_1})\dots(1-\alpha_{\ell_{m-1}})(1-\alpha_1)^{M-m+1}}\right)\frac{(1-\alpha_{j+1})}{(1-\alpha_j)}u'_+\big(f(b_j)\big)\\ &= \frac{(1-\alpha_{j+1})}{(1-\alpha_j)}u'_+\big(f^M(f(b_j))\big)\\ &> u'_+\big(f^M(f(b_j))\big). \end{aligned}$$

In view of (15), this contradicts the differentiability of $u(\cdot)$ on $(0,\gamma)$. The proof of Theorem 3.5 is now complete.

6.4. *Proof of Proposition* 5.1. Assume the hypotheses of Theorem 3.5 and suppose that

$$u(x)-u(f(x))=c \text{ for all } x>0,$$

for some $c>0$ and strictly increasing and concave $u\colon \mathbf{R}_{++}\to\mathbf{R}$. We can use (12) to conclude that

$$(16)\qquad u'_-(f(b_2)) \geq \frac{(1-\alpha_3)}{(1-\alpha_2)}u'_+(f(b_2)).$$

Now since $f^2(b_2) < f(b_2)$, by concavity of $u(\cdot)$,

$$u'_+(f^2(b_2)) \geq u'_-(f(b_2)). \tag{17}$$

By (16) and (17),

$$u'_+(f^2(b_2)) \geq \frac{(1-\alpha_3)}{(1-\alpha_2)} u'_+(f(b_2)). \tag{18}$$

But by (8) we have $f(b_2) < b_1$, and thus, by proceeding as in (10) we have

$$u'_+(f(b_2)) = u'_+(f^2(b_2))(1-\alpha_1),$$

and combining this with (18),

$$\frac{1}{1-\alpha_1} u'_+(f(b_2)) = u'_+(f^2(b_2)) \geq \frac{(1-\alpha_3)}{(1-\alpha_2)} u'_+(f(b_2)).$$

This contradicts (9) and establishes the result.

7. CONCLUSION

Although the doctrine of ability to pay, and in particular, the principle of equal sacrifice, has a traditionally venerable position in the theory of public finance, it is a widely held belief that the general theory is largely inconclusive with regard to the problem of tax design. One of the major reasons for this position is the brief demonstration of Samuelson (1947, p. 227) that the principle of equal sacrifice does *not* imply the progressivity of the personal income tax, and, in effect, that the related insights of a number of classical economists, like J. S. Mill, A. J. Cohen Stuart, H. Sidgwick, were simply wrong.

We started the present paper by observing that Samuelson's demonstration is incomplete. Then, departing slightly from the conventional framework, we restricted our attention to the domain of piecewise linear tax functions, and noted that this departure brings us closer to the actual discourse of taxation. Our findings show that Mill's original insight was right after all in that the equal sacrifice doctrine does imply tax progressivity. In fact, the progressivity implications of the doctrine is stronger: the principle implies marginal rate progressivity of the income tax schedule, except in some pathological cases (which are shown to be rather negligible in the Appendix). The converse of this statement is also true, and thus, the principle of equal sacrifice, in essence, *characterizes* marginal rate progressivity; a result which is in sharp contrast with the common contention.

We conclude by noting that the only OECD countries which can readily be spotted by our results as violating the principle of equal sacrifice in the period 1988–1991 are the United States and Turkey. It is interesting to note that we could not have said the same for the U.S. federal personal statutory income taxation scheme prior to the Tax Reform Act of 1986.

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APPENDIX

In Section 3.3, we have refined the class of all admissible tax functions by eliminating a subset which we qualified as 'pathological' throughout our study. Since our results are considerably stronger in this refined set $\mathscr{T}_0$, the size of the eliminated set of tax functions (which we no longer view as admissible) is important. In this Appendix we shall formally argue that the eliminated class is negligible in a measure-theoretic sense. Put precisely, for any given positive integer N, we shall make $\mathscr{T}(N)$ a measure space in a natural way, and then show that $\mathscr{T}(N)\setminus\mathscr{T}_0$ is of measure zero.

Let $N \geq 2$ be an arbitrary integer. Notice that any member of $\mathscr{T}(N)$ is completely determined by $3N-2$ real-valued parameters. Indeed, the function σ_N: $\mathscr{T}(N) \to \mathbf{R}^{3N-2}$ defined as

$$\sigma_N(t) := (b_1, b_2, \dots, b_{N-1}, \alpha_1, \alpha_2, \dots, \alpha_N, \theta_1, \theta_2, \dots, \theta_{N-1})$$

where $t(\cdot)$ is given by (3), (4) and (5), is a one-to-one function. Therefore, via $\sigma_N(\cdot)$, we can embed $\mathscr{T}(N)$ into $\mathbf{R}^{3N-2}$ and evaluate the 'size' of a given subset of $\mathscr{T}(N)$ by the Lebesgue measure of the image of this subset under $\sigma_N(\cdot)$. To formalize this idea, let $\mathscr{B}_n$ be the Borel σ-field in $\mathbf{R}^n$, $n \geq 2$, and let $\mu_n(\cdot)$ denote the n-dimensional Lebesgue measure. Define

$$\Sigma_N := \{\sigma_N^{-1}(A) : A \in \mathscr{B}_{3N-2}\}$$

and

$$\lambda_N(T) = \mu_{3N-2}(\sigma_N(T)) \text{ for any } T \in \Sigma_N. \tag{19}$$

One can easily check that Σ_N in $\mathscr{T}(N)$ is a σ-field and $\lambda_N(\cdot)$ is a measure on Σ_N. Therefore, $(\mathscr{T}(N), \Sigma_N, \lambda_N)$ defines a measure space.

Our previous claim that restricting attention to $\mathscr{T}_0$ as opposed to $\mathscr{T}$ comes with only negligible loss of generality, can then be formalized as follows:

THEOREM A1. *For any* $N \geq 2$, $\lambda_N(\mathscr{T}(N)\setminus\mathscr{T}_0) = 0$.

For the sake of brevity, we shall prove this proposition only for the case $N = 3$. But we note that the argument is quite elementary and the generalization of it is rather straightforward.

Let $t \in \mathscr{T}(3)\setminus\mathscr{T}_0$. Such a tax function $t(\cdot)$ can be written as

$$t(x) = \begin{cases} \alpha_1 x, & \text{if } 0 \leq x \leq b_1 \\ \alpha_2 x + \theta_1 & \text{if } b_1 < x \leq b_2 \\ \alpha_3 x + \theta_2 & \text{if } b_2 < x \end{cases}$$

where $0 < \alpha_i < 1$, $i = 1,2,3$, $\alpha_1 \neq \alpha_2 \neq \alpha_3$, $\theta_i = t(b_i) - \alpha_{i+1} b_i$, $i = 1,2$ and

$$f^n(b_2) = b_1 \text{ for some } n \geq 1, \tag{20}$$

with $f(x)=x-t(x)$ for all $x>0$. Now, α_1 and α_3 can clearly take any value in $(0,1)$, and b_1 and b_2 can take any value in $(0,\infty)$ as long as $b_1<b_2$. But once α_1, b_1, and b_2 are specified, by the continuity of $t(\cdot)$ at b_1 and by (20), the set of all possible α_2 and θ_1 values is given by

$$A(\alpha_1,b_1,b_2) := \bigcup_{n\geq 1} \{(\alpha_2,\theta_1)\in(0,1)\times\mathbf{R}: \theta_1=\alpha_1 b_1-\alpha_2 b_1 \text{ and } f^n(b_2)=b_1\}.$$

For any given α_1, b_1, b_2, and $n\geq 1$, by the fundamental theorem of algebra, there are at most n solutions of the following polynomial equation system:

$$\theta_1=\alpha_1 b_1-\alpha_2 b_1$$

$$f^n(b_2) := (1-\alpha_2)^n b_2 - \sum_{j=0}^{n-1}(1-\alpha_2)^j\theta_1 = b_1.$$

Therefore, we may conclude that, for any given α_1, b_1, b_2, $A(\alpha_1,b_1,b_2)$ is a countable set, and hence

$$\mu_2(A(\alpha_1,b_1,b_2))=0. \tag{21}$$

Finally, we note that once b_2, α_2 and θ_1 are specified, then θ_2 can take any value in $((\alpha_2-1)b_2+\theta_1,\ \alpha_2 b_2+\theta_1)$.

By the above analysis, Fubini's theorem, and (19), we have

$$\lambda_3(\mathscr{T}(3)\setminus\mathscr{T}_0)=\int_0^\infty\int_{b_1}^\infty\int_0^1\int_0^1\iint_{A(\alpha_1,b_1,b_2)}\int_{(\alpha_2-1)b_2+\theta_1}^{\alpha_2 b_2+\theta_1} d\theta_2\, d(\alpha_2,\theta_1)\, d\alpha_3\, d\alpha_1\, db_2\, db_1$$

Therefore, by (21), $\lambda_3(\mathscr{T}(3)\setminus\mathscr{T}_0)=0$.

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